SOLUTIONS TO EXAM 1, MATH 10550

1. Compute

$$\lim_{x \to -1^-} \frac{x^2 - 1}{x^2 + 2x + 1}.$$

Answer: ∞.

Solution: $\lim_{x\to -1^-} \frac{x^2-1}{x^2+2x+1} = \lim_{x\to -1^-} \frac{(x-1)(x+1)}{(x+1)(x+1)} = \lim_{x\to -1^-} \frac{x-1}{x+1} = \infty.$

2. All the vertical asymptotes of the function $f(x) = \dfrac{x^2 - 1}{x^3 - 9x}$ are at

Answer: $x = 0$ and $x = \pm 3$

Solution: Write $f(x) = \dfrac{g(x)}{h(x)}$ where $g(x) = x^2 - 1$ and $h(x) = x^3 - 9x$. Since $f(x)$ is a rational function, its vertical asymptotes are among those lines $x = a$ where $a \in \mathbb{R}$ satisfies $h(a) = 0$. Since $h(x) = x^3 - 9x = x(x^2 - 9) = x(x-3)(x+3)$, we have $a = 0, 3, -3$. Finally, we must check that for each $a \in \{0, 3, -3\}$, one of the one-sided limits $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$ is $\pm\infty$. For $a = 0$, $\lim_{x\to 0^+} f(x) = +\infty$. For $a = 3$, $\lim_{x\to 3^+} f(x) = +\infty$. For $a = -3$, $\lim_{x\to -3^+} f(x) = +\infty$. Thus the asymptotes of $f(x)$ are the lines $x = 0$, $x = 3$, and $x = -3$.

3. For what value a is the function f given by

$$f(x) = \begin{cases} \frac{\sqrt{9+x^2}-3}{x^2} & x \neq 0 \\ a & x = 0 \end{cases}$$

continuous everywhere?

Answer: $\dfrac{1}{6}$

Solution: For $f(x)$ to be continuous at $x = 0$, we must choose a so that:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sqrt{9+x^2}-3}{x^2} = f(0) = a.$$

So we have

$$a = \lim_{x\to 0} \frac{\sqrt{9+x^2}-3}{x^2} = \lim_{x\to 0} \frac{(\sqrt{9+x^2}-3)}{x^2} \frac{(\sqrt{9+x^2}+3)}{(\sqrt{9+x^2}+3)} = \lim_{x\to 0} \frac{9+x^2-9}{x^2(\sqrt{9+x^2}+3)} =$$

$$\lim_{x\to 0} \frac{x^2}{x^2(\sqrt{9+x^2}+3)} = \lim_{x\to 0} \frac{1}{(\sqrt{9+x^2}+3)} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}.$$

4. Find $f'(2)$ if
$$f(x) = 4\sqrt{x+2} - \frac{16}{\sqrt{x+2}}.$$
Answer: 2

Solution: First write $f(x)$ as a rational function:
$$f(x) = 4\sqrt{x+2} - \frac{16}{\sqrt{x+2}} = \frac{4(x+2)-16}{\sqrt{x+2}} = \frac{4x-8}{\sqrt{x+2}}.$$
Using the Quotient Rule, we have the formula
$$f'(x) = \frac{4(\sqrt{x+2}) - \frac{4x-8}{2\sqrt{x+2}}}{x+2}.$$
Evaluating at $x = 2$ yields
$$f'(2) = \frac{4\sqrt{4} - \frac{4\cdot 2-8}{\sqrt{4}}}{2+2} = \frac{4\cdot 2 - 0}{4} = 2.$$

5. Find the equation of the tangent line to the curve $y = 6\sqrt{x} + 2$ at $x = 9$.
Answer: $y = x + 11$

Solution: $y' = \dfrac{6}{2\sqrt{x}} = \dfrac{3}{\sqrt{x}}$. To find the slope m of the tangent line, we evaluate y' at $x = 9$:
$$m = \frac{3}{\sqrt{9}} = 1.$$
When $x = 9$, $y = 20$, so to find the tangent line, we use the point-slope formula with slope $m = 1$ and point $(9, 20)$:
$y - 20 = 1(x - 9)$, or $y = x + 11$.

6. Find the derivative of $f(x) = (3 + x^3)^{2/3}$.
Answer: $2x^2(3 + x^3)^{-1/3}$

Solution: We apply chain rule with $g(x) = x^{2/3}$, $h(x) = (3 + x^3)$. So $f(x) = g(h(x))$, and it follows that
$$\begin{aligned} f'(x) &= g'(h(x)) \cdot h'(x) \\ &= \frac{2}{3}(3 + x^3)^{-1/3} \cdot 3x^2 \\ &= 2x^2(3 + x^3)^{-1/3}. \end{aligned}$$

7. Compute the derivative of

$$f(x) = \frac{x^2 + \cos x}{x + \cos^2 x}.$$

Answer: $\dfrac{(2x - \sin x)(x + \cos^2 x) - (1 - 2\sin x \cos x)(x^2 + \cos x)}{(x + \cos^2 x)^2}$

Solution: Using quotient rule with $f(x) = \frac{g(x)}{h(x)}$, where $g(x) = x^2 + \cos x$ and $h(x) = x + \cos^2 x$, we obtain

$$\begin{aligned} f'(x) &= \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2} \\ &= \frac{(x + \cos^2 x)(2x - \sin x) - (x^2 + \cos x)(1 - 2\cos x \sin x)}{(x + \cos^2 x)^2}. \end{aligned}$$

8. If $f(x) = \cos(x^2)$, find $f''(x)$.
Answer: $f''(x) = -2\sin(x^2) - 4x^2\cos(x^2)$

Solution: We apply the chain rule to obtain the first derivative:

$$\begin{aligned} f'(x) &= -\sin(x^2) \cdot 2x \\ &= -2x\sin(x^2). \end{aligned}$$

Next, we differentiate the first derivative to obtain $f''(x)$:

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left[f'(x)\right] \\ &= \frac{d}{dx}\left[-2x\sin(x^2)\right] \\ &= \frac{d}{dx}[-2x] \cdot (\sin(x^2)) + (-2x)\frac{d}{dx}\left[\sin(x^2)\right] && \text{(Product Rule)} \\ &= -2\sin(x^2) - 2x\left(\cos(x^2) \cdot \frac{d}{dx}[x^2]\right) && \text{(Chain Rule)} \\ &= -2\sin(x^2) - 2x\cos(x^2) \cdot 2x \\ &= -2\sin(x^2) - 4x^2\cos(x^2). \end{aligned}$$

9. Compute

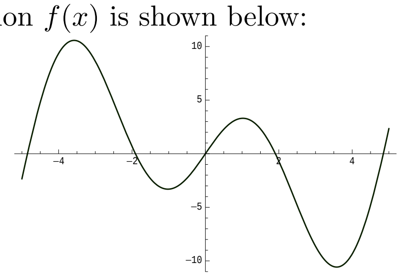
$$\lim_{x\to 0}\frac{\sin(4x)}{\tan(9x)}.$$

Answer: $\dfrac{4}{9}$

Solution: We note that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ and evaluate the limit as follows:

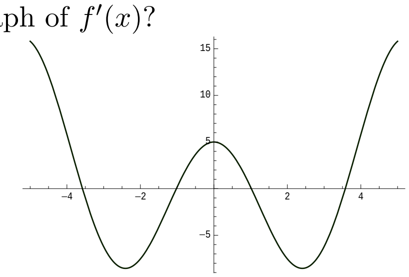
$$\begin{aligned}
\lim_{x\to 0}\frac{\sin(4x)}{\tan(9x)} &= \lim_{x\to 0}\frac{\sin(4x)}{\frac{\sin(9x)}{\cos(9x)}}\\
&= \lim_{x\to 0}\frac{\sin(4x)\cos(9x)}{\sin(9x)}\\
&= \lim_{x\to 0}\frac{4\,\cancel{x}\cdot\frac{\sin(4x)}{4x}\cos(9x)}{9\,\cancel{x}\cdot\frac{\sin(9x)}{9x}}\\
&= \lim_{x\to 0}\frac{4\cdot\frac{\sin(4x)}{4x}\cos(9x)}{9\cdot\frac{\sin(9x)}{9x}}\\
&= \frac{4\cdot\lim_{x\to 0}\frac{\sin(4x)}{4x}\cdot\lim_{x\to 0}\cos(9x)}{9\cdot\lim_{x\to 0}\frac{\sin(9x)}{9x}}\\
&= \frac{4\cdot 1\cdot 1}{9\cdot 1}\\
&= \frac{4}{9}.
\end{aligned}$$

10. The graph of the function $f(x)$ is shown below:



Which of the following gives the graph of $f'(x)$?

Answer:



Solution: The function $f(x)$ is increasing on the following (approximate) intervals: $(-\infty,-3.6),(-1,1)$, and $(3.6,\infty)$. The graph of $f'(x)$ must be positive on these intervals. The function $f(x)$ is decreasing on the intervals $(-3.6,-1)$ and $(1,3.6)$. This means that $f'(x)$ must be negative on these intervals. $f'(x)$ is zero when $x=-3.6,-1,1,3.6$.

The above graph satisfies these requirements. Estimating slopes of tangent lines gives us $f'(-4) \approx 5, f'(2) \approx -7, f(0) \approx 5, f(2) \approx -8$ and $f(4) \approx 6$ which also agrees with the given graph. Note that the requirement that $f(x)$ is increasing on the interval $(-\infty, -3.6)$ eliminates 3 of the 4 graphs.

11. Show that there are at least two solutions of the equation

$$x^4 = 6x - 1.$$

Be sure to check the hypotheses of any theorem you might use.

Solution: We must show that the equation

$$x^4 - 6x + 1 = 0$$

has two solutions. Let $f(x) = x^4 - 6x + 1$. Since f is a polynomial, f is continuous everywhere, so we may use the Intermediate Value Theorem.
First, $f(0) = 1$ and $f(1) = -4$, so since $f(0) > 0 > f(1)$, we conclude by the Intermediate Value Theorem that f has at least one zero in the interval $(0, 1)$.
Similarly, since $f(1) = -4 < 0$ and $f(2) = 5 > 0$, we conclude by the Intermediate Value Theorem that f has at least one zero in the interval $(1, 2)$.
Finally, since the intervals $(0, 1)$ and $(1, 2)$ are disjoint, f has at least two zeros.

12. Find the derivative of

$$y = \frac{1}{\sqrt{x} + 1}$$

using the definition of the derivative.

Solution:

$$\begin{aligned}
y' &= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}+1} - \frac{1}{\sqrt{x}+1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x} + 1 - \sqrt{x+h} - 1}{h(\sqrt{x+h}+1)(\sqrt{x}+1)} \\
&= \lim_{h\to 0} \frac{(\sqrt{x} - \sqrt{x+h})}{h(\sqrt{x+h}+1)(\sqrt{x}+1)} \frac{(\sqrt{x} + \sqrt{x+h})}{(\sqrt{x} + \sqrt{x+h})} \\
&= \lim_{h\to 0} \frac{x - (x+h)}{h(\sqrt{x+h}+1)(\sqrt{x}+1)(\sqrt{x} + \sqrt{x+h})} \\
&= \lim_{h\to 0} \frac{-h}{h(\sqrt{x+h}+1)(\sqrt{x}+1)(\sqrt{x} + \sqrt{x+h})} \\
&= \lim_{h\to 0} \frac{-1}{(\sqrt{x+h}+1)(\sqrt{x}+1)(\sqrt{x} + \sqrt{x+h})} \\
&= \frac{-1}{(\sqrt{x}+1)(\sqrt{x}+1)(\sqrt{x} + \sqrt{x})} \\
&= \frac{-1}{2\sqrt{x}(\sqrt{x}+1)^2}.
\end{aligned}$$

13. At what point(s) on the ellipse $3x^2+y^2 = 21$ is the tangent line at that point parallel to the straight line $y = -2x + 6$?

Solution: We first use implicit differentiation to find $\frac{dy}{dx}$. Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left(3x^2 + y^2\right) = \frac{d}{dx}(21).$$

Using the product rule on the left-hand side, we get

$$6x + 2y\frac{dy}{dx} = 0.$$

We solve for $\frac{dy}{dx}$:

$$\begin{aligned} 6x + 2y\frac{dy}{dx} &= 0 \\ \Rightarrow 2y\frac{dy}{dx} &= -6x \\ \Rightarrow \frac{dy}{dx} &= -\frac{3x}{y}. \end{aligned}$$

The tangent line to the ellipse is parallel to the straight line $y = -2x + 6$ whenever it has slope -2, or, equivalently, when $\frac{dy}{dx} = -2$. Setting $\frac{dy}{dx} = -\frac{3x}{y} = -2$, we obtain the relation $-3x = -2y$, which reduces to $y = \frac{3x}{2}$. So we need to find all points on the ellipse where $y = \frac{3x}{2}$. Substituting $\frac{3x}{2}$ for y into the equation for the ellipse, we obtain

$$\begin{aligned} 3x^2 + \frac{9x^2}{4} &= 21 \\ &\Rightarrow \frac{21x^2}{4} = 21 \\ &\Rightarrow x^2 = 4 \\ &\Rightarrow x = \pm 2 \end{aligned}$$

Since $y = \frac{3x}{2}$, the two points at which the tangent line to the curve has slope -2 are $(2, 3)$ and $(-2, -3)$.